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Super Calabi-Yau's and special Lagrangians

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Abstract: We apply mirror symmetry to the super Calabi-Yau manifold $\mathbf{CP}^{(n|n+1)}$ and show that the mirror can be recast in a form which depends only on the superdimension and which is reminiscent of a generalized conifold. We discuss its geometrical properties in comparison to the familiar conifold geometry. In the second part of the paper examples of special-Lagrangian submanifolds are constructed for a class of super Calabi-Yau's. We finally comment on their infinitesimal deformations.

Keywords: Sigma Models, Supersymmetry and Duality, Differential and Algebraic Geometry.

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Contents

1. Introduction

Recent interest in super Calabi-Yau manifolds comes from the duality between the topological B model on $\mathbf{CP}^{(3|4)}$ and *perturbative* super Yang-Mills. This surprising connection has led to a new understanding of perturbative Yang-Mills [2]. For a review see [3] and [4]. See also [5 − 9] for a partial list of further developments. Even though this duality can be seen as an extremely interesting counterpart of the AdS/CFT correspondence, it has also given a new impetus to the study of purely geometrical properties of super Calabi-Yau manifolds. See for instance [10, 11] for novel results in this direction.

Super Calabi-Yau manifolds provide an interesting arena for studying topological strings. One remarkable conjecture is that the topological A model on $\mathbf{CP}^{(3|4)}$ is equivalent to the topological B model on a quadric inside the (super)ambi-twistor space $\mathbf{CP}^{(3|3)} \times \mathbf{CP}^{(3|3)}$ [12, 6]. A crucial ingredient in this conjecture is mirror symmetry. The importance of supermanifolds in the context of mirror symmetry was fully appreciated for the first time in [13]: Landau-Ginzburg models which are mirror to rigid Calabi Yau's[1] can be given a geometrical interpretation as sigma models with supermanifolds as target space. The modern language for studying mirror symmetry for toric supermanifolds has been systematized in [14]. For other related works see [16 − 18]. In the first part of the paper we will apply mirror symmetry to the super Calabi-Yau $\mathbf{CP}^{(n|n+1)}$ and show that the mirror can be recast in a form which is reminiscent of a generalized conifold. The mirror depends only on the superdimension of the supermanifold, i.e. on the difference of bosonic and fermionic dimensions. We then discuss its geometrical properties in comparison with the usual, bosonic, conifold geometry.

[1] A Calabi-Yau is rigid when it does not have complex structure moduli.

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In Calabi-Yau compactifications special Lagrangian submanifolds are particularly important because they are supersymmetric cycles, known as A branes since they preserve the A model topological charge. It is interesting to see whether special-Lagrangian submanifolds can be constructed inside Calabi-Yau supermanifolds. In the second part of the paper examples of special-Lagrangians are constructed for a class of super Calabi-Yau's in a similar spirit to what done in [20] for local Calabi-Yau's.

Apart from those already mentioned, there other reasons of interests in super Calabi-Yau's. The most prominent is, perhaps, the fact that, as far as the topological A model is concerned, certain *compact* bosonic Calabi-Yau's are equivalent to (toric) super Calabi-Yau's [19]. An example is the A model on the classic Calabi-Yau quintic in $\mathbf{CP}^4$ which is equivalent to the A model on the super-projective Calabi-Yau space $\mathbf{CP}(1,1,1,1,1|5)$. In [20, 21] open string instanton corrections to the worldvolume superpotential for some non-compact special Lagrangian branes have been derived for a class of non-compact Calabi-Yau's using mirror symmetry. We can then speculate that using similar techniques, and in view of the above remarks, the study of Lagrangian submanifolds in super Calabi Yau's could maybe help in performing the superpotential computation in the notoriously difficult compact Calabi-Yau case.

The organization of the paper is as follows: We begin by reviewing the relevant aspects of mirror symmetry in section 2; In section 3 we apply mirror symmetry to $\mathbf{CP}^{(n|n+1)}$ and discuss the mirror "super-conifold" geometry which arises in the dual theory. In section 4 we review the construction of non-compact special Lagrangian in toric CY manifolds. This construction is suitably extended to the supermanifold case in the next section; In the last section we finally comment on the moduli space of infinitesimal deformations of (super)special-Lagrangians.

2. Gauged linear sigma model and mirror symmetry

In this section we review the proof of mirror symmetry for local Calabi-Yau manifolds [22]. The proof consists in showing the equivalence of a gauged linear sigma model and a dual Landau-Ginzburg theory. The gauged linear sigma model reduces in the low energy limit to a non-linear sigma model on the Calabi-Yau manifold [23].[2] We work in 1+1 dimensions where we study the following $(2,2)$ supersymmetric gauge theory

$$\mathcal{L} = \int d^4\theta \left(\sum_i \bar{\Phi}^i e^{2Q_i V} \Phi^i - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) - \frac{1}{2} \int d^2\tilde{\theta}\, t\Sigma + c.c. \tag{2.1}$$

The chiral fields Φ^i have charges Q_i under the U(1) gauge group with vector superfield V. The twisted chiral field strength is $\Sigma = \bar{D}_+ D_- V$, $t = r - i\theta$ is the complexified Fayet-Iliopoulos parameter and $d^2\tilde{\theta}$ is the twisted chiral superspace measure $d\theta^+ d\bar{\theta}^-$. In the low-energy limit $r_0 \gg 1$ the theory is equivalent to a non-linear sigma model on the toric manifold

$$\left\{ \sum_{i=1}^{N} Q_i |\Phi^i|^2 = r_0 \right\} / \, \mathrm{U}(1) \tag{2.2}$$

[2] See also [24] for a discussion of gauged linear sigma models on supermanifolds.

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If $\sum_{i=1}^{N} Q_i = 0$ the bare real F.I. parameter r_0 does not renormalize. The parameter t is identified with the complexified Kähler parameter of the sigma model. The case $\sum_{i=1}^{N} Q_i = 0$ corresponds to a local Calabi-Yau space.

Let us consider the"enlarged" Lagrangian

$$\mathcal{L} = \int d^4\theta \left(e^{2QV+B} - \frac{1}{2}(Y + \bar{Y})B \right) \tag{2.3}$$

where B is a real superfield and Y a twisted chiral field, $\bar{D}_+ Y = D_- Y = 0$, whose imaginary part has period 2π. Rewriting the superspace measure as $d^4\theta = d\theta^+ d\bar{\theta}^- \, D_- \bar{D}_+$, the field equation for Y

$$\frac{\delta}{\delta Y} \int d\theta^+ d\bar{\theta}^- \, Y(D_- \bar{D}_+ B) = 0, \tag{2.4}$$

yields

$$D_- \bar{D}_+ B = 0. \tag{2.5}$$

This equation enforces the decomposition

$$B = \psi + \bar{\psi}, \tag{2.6}$$

where ψ is a chiral superfield. Inserting this expression in (2.3) the Lagrangian becomes

$$\mathcal{L} = \int d^4\theta \, e^{2QV+\psi+\bar{\psi}} = \int d^4\theta \; \bar{\Phi} e^{2QV} \Phi \tag{2.7}$$

where we have introduced another chiral field $\Phi = e^{\psi}$.

Alternatively, we can first integrate out B in (2.3) obtaining

$$B = -2QV + \log \left(\frac{Y + \bar{Y}}{2} \right). \tag{2.8}$$

After inserting this result back in the Lagrangian, this yields

$$\mathcal{L} = \int d^4\theta \left(-\frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) + QV(Y + \bar{Y}) \right) \tag{2.9}$$

which, using $\Sigma = \bar{D}_+ D_- V$, can be rewritten as

$$\mathcal{L} = \int d^4\theta \left(-\frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) + \int d^2\tilde{\theta} Q\Sigma Y + c.c. \tag{2.10}$$

Therefore we have shown that the Lagrangian

$$\mathcal{L} = \int d^4\theta \left(\bar{\Phi} e^{2QV} \Phi - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) - \frac{1}{2} \int d^2\tilde{\theta} \, t\Sigma + c.c. \tag{2.11}$$

is classically dual to

$$\mathcal{L} = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma}\Sigma - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) + \frac{1}{2} \int d^2\tilde{\theta} \Sigma(QY - t) + c.c. \tag{2.12}$$

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In the duality the chiral superfield Φ is exchanged with a *twisted* chiral superfield Y. Comparing the different expressions (2.6) and (2.8) for B we obtain

$$\text{Re}Y = 2\bar{\Phi}e^{2QV}\Phi. \tag{2.13}$$

In the Wess-Zumino gauge this relation implies that the lowest components φ and y of the chiral and twisted fields satisfy $\text{Re}\, y = 2|\varphi|^2$. If we generalize the discussion to a gauge theory with n chiral fields Φ_i, we get a dual superpotential $\tilde{W} = \sum_i (Q_i Y_i - t)\Sigma$. At the quantum level, non-perturbative instanton corrections modify the dual twisted superpotential into $\tilde{W} = \sum_i (Q_i Y_i - t)\Sigma + e^{-Y_i}$. Integrating out Σ gives

$$\sum_i^n Q_i Y_i = t \tag{2.14}$$

which is the dual version of the D-term constraint of the original gauge theory.

As an example we can consider the gauged linear sigma model with chiral fields $(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ and charges $(1, 1, -1, -1)$. In the low-energy limit this theory is equivalent to a non-linear sigma model on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. The lowest components of the fields with positive charge parametrize the $\mathbf{CP}^1$ in the base, while the fields with negative charge span the non-compact fibers. The T dual-mirror theory is a Landau-Ginzburg model with dual fields Y_i that satisfy

$$\text{Re}Y_i = |\Phi_i|^2 \tag{2.15}$$

and superpotential $\tilde{W} = \sum_{i=1}^4 e^{-Y_i}$, subject to the constraint

$$Y_1 + Y_2 - Y_3 - Y_4 = t. \tag{2.16}$$

The complex Fayet-Iliopoulos parameter is the complexified Kähler class of the $\mathbf{CP}^1$ in the non linear sigma model. The Landau Ginzburg path integral is

$$\int dY_i\, \delta(Y_1 + Y_2 - Y_3 - Y_4 - t)\, \exp\left(\sum_{i=1}^4 e^{-Y_i}\right) \tag{2.17}$$

Solving the constraint by integrating out Y_1 and defining $y_i = \exp(-Y_i)$ yields

$$\int \prod_{i=2}^4 \frac{dy_i}{y_i}\ \exp\left(y_2 + y_3 + y_4 + \frac{y_3 y_4}{y_2} e^{-t}\right). \tag{2.18}$$

Redefining $\tilde{y}_2 = y_2/y_4$, $\tilde{y}_3 = y_3/y_4$ and introducing auxiliary variables u, v in $\mathbf{C}$ so that

$$\frac{1}{y_4} = \int dudv\, e^{uvy_4} \tag{2.19}$$

we can rewrite (2.18) as

$$\begin{aligned} &\int \frac{d\tilde{y}_2}{\tilde{y}_2}\frac{d\tilde{y}_3}{\tilde{y}_3} dy_4 dudv\ \exp\left(\tilde{y}_2 y_4 + \tilde{y}_3 y_4 + y_4(uv+1) + \frac{\tilde{y}_3 y_4}{\tilde{y}_2} e^{-t}\right) \\ = &\int \frac{d\tilde{y}_2}{\tilde{y}_2}\frac{d\tilde{y}_3}{\tilde{y}_3} dudv\, \delta\left(\tilde{y}_2 + \tilde{y}_3 + uv + 1 + \frac{\tilde{y}_3}{\tilde{y}_2} e^{-t}\right), \end{aligned} \tag{2.20}$$

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where in the last step y_4 has been treated as a Lagrange multiplier and integrated out. Therefore the mirror geometry, in the patch $y_4 = 1$, can be regarded as the Calabi-Yau hypersurface

$$uv = \tilde{y}_2 + \tilde{y}_3 + \frac{\tilde{y}_3}{\tilde{y}_2} e^{-t}, \tag{2.21}$$

after a suitable redefinition of u and v. Mirror symmetry than implies that the topological A model on the resolved conifold is equivalent to the B model on the mirror Calabi-Yau. Note that the Kähler parameter t of the initial theory gets exchanged with the complex parameter e^{-t} of the mirror.

3. Superconifold geometries

Our prototype for a supermanifold is the superprojective space $\mathbf{CP}^{(n|m)}$ with bosonic and fermionic coordinates z^i, ψ^A subject to the identification

$$(z^1, \ldots, z^{n+1}|\psi^1, \ldots, \psi^m) \sim \lambda(z^1, \ldots, z^{n+1}|\psi^1, \ldots, \psi^m) \tag{3.1}$$

where λ is a complex number different from zero. The *superdimension* is the difference of bosonic and fermionic dimensions. In this case $sdim_{\mathbf{CP}^{(n|m)}} = n - m$. It is straightforward to generalize this construction to weighted superprojective spaces like $\mathbf{CP}(Q_1, \ldots, Q_n|P_1, \ldots, P_m)$, where Q_i and P_i are respectively the charges of the bosonic and fermionic coordinates under the $C^\star$ action. To find a simple example of super Calabi-Yau we may start from the supermanifold $\mathbf{C}^{(n+1|m)}$ with holomorphic measure $\Omega_0 = dz^1 \wedge \ldots \wedge dz^{n+1} \bigotimes \partial_{\psi^1} \ldots \partial_{\psi^m}$. The form Ω_0 descends to a holomorphic form Ω on the quotient space $\mathbf{CP}(Q_1, \ldots, Q_{n+1}|P_1, \ldots, P_m)$ if the super Calabi-Yau condition

$$\sum_{i=1}^{n+1} Q_i - \sum_{A=1}^{m} P_A = 0 \tag{3.2}$$

is satisfied. The minus sign in front of P_A is due to the fact that ψ and ∂_ψ have opposite charges do the Berezin integration rule $\int d\psi\, \psi = 1$. The condition expressed by eq. (3.2) amounts to say that the Berezinian line bundle of the supermanifold is trivial.

Let us briefly review how mirror symmetry generalizes to supermanifolds. We start with a U(1) gauged linear sigma model with bosonic and fermionic chiral fields ϕ^i, ψ^A and charges Q_i, P_A respectively. The D term equation is then

$$\sum_i Q_i \left|\phi^i\right|^2 + \sum_A P_A \left|\psi^A\right|^2 = r \tag{3.3}$$

The space of vacua is the supermanifold obtained by dividing (3.3) by the U(1) group. The dual fields which appear in the mirror theory are related to ϕ^i, ψ^A as follows

$$\mathrm{Re} Y^i = |\phi^i|^2 \tag{3.4}$$

$$\mathrm{Re} X^A = -|\psi^A|^2 \tag{3.5}$$

This is the usual correspondence modulo the fact that X^A, dual to the fermionic field ψ^A, picks an additional minus sign. To guarantee that the original and the mirror supermanifolds have the same superdimension, we need to add a couple of fermionic fields η, χ to bosonic field X. The D term constraint (3.3) is mirrored into

$$\sum_i Q_i Y^i - \sum_A P_A X^A = t \tag{3.6}$$

where t is the complexified Kähler parameter. The superpotential for the mirror Landau Ginzburg description is similar to the bosonic case

$$W = \sum_{i=1} e^{-Y^i} + \sum_{A=1} e^{-X^A} \left(1 + \eta^A \chi^A\right) \tag{3.7}$$

modulo the presence of the additional contribution $\sum_{A=1} e^{-X^A} \eta^A \chi^A$ for the fermionic fields. It is intended that the fields satisfy the D term constraint(3.6). Using this technique, it has been shown [14] that the mirror of $\mathbf{CP}^{(3|4)}$ is a super Calabi-Yau hypersurface

$$\sum_{i=1}^{3} x_i y_i + x_i + 1 + e^t y_1 y_2 y_3 + \eta_i \chi_i = 0. \tag{3.8}$$

In the limit $t \to -\infty$, eq. (3.8) can be thought as a quadric in a patch of $\mathbf{CP}^{(3|3)} \times \mathbf{CP}^{(3|3)}$ with local inhomogeneous coordinates (x_i, η_i) and (y_i, χ_i).

We now apply mirror symmetry to the supermanifold $\mathbf{CP}^{(n|n+1)}$. The path integral for the mirror Landau Ginzburg model is

$$\int \prod_{i=1}^{n+1} dY_i dX_i d\eta_i d\chi_i \delta \left(\sum_{i=1}^{n+1} Y_i - \sum_{i=1}^{n+1} X_i - t\right) \exp\left(\sum_{i=1}^{n+1} e^{-Y_i} + \sum_{i=1}^{n+1} e^{-X_i} \left(1 + \eta_i \chi_i\right)\right) \tag{3.9}$$

Solving the delta function constraint by integrating out X_1 yields

$$\int \prod_{i=1}^{n+1} dY_i d\eta_i d\chi_i \prod_{j=2}^{n+1} dX_j \tag{3.10}$$

$$\exp\left(\sum_{i=1}^{n+1} e^{-Y_i} + e^t \prod_{i=1}^{n+1} e^{-Y_i} \prod_{j=2}^{n+1} e^{X_j} \left(1 + \eta_1 \chi_1\right) + \sum_{i=2}^{n+1} e^{-X_i} \left(1 + \eta_i \chi_i\right)\right) \tag{3.11}$$

Now we integrate over the fermionic fields η_1, χ_1

$$\int \prod_{i=1}^{n+1} dY_i e^{-Y_i} \prod_{j=2}^{n+1} dX_j e^{X_j} d\eta_j d\chi_j \tag{3.12}$$

$$\exp\left(\sum_{i=1}^{n+1} e^{-Y_i} + e^t \prod_{i=1}^{n+1} e^{-Y_i} \prod_{j=2}^{n+1} e^{X_j} + \sum_{i=2}^{n+1} e^{-X_i} \left(1 + \eta_i \chi_i\right)\right) \tag{3.13}$$

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We did not include an irrelevant overall factor e^{-t}. We integrate in a similar way over all the remaining fermionic coordinates except η_{n+1}, χ_{n+1} obtaining

$$\int \prod_{i=1}^{n+1} dY_i e^{-Y_i} \prod_{j=2}^{n+1} dX_j e^{X_{n+1}} d\eta_{n+1} d\chi_{n+1}$$
$$\exp\left(\sum_{i=1}^{n+1} e^{-Y_i} + e^t \prod_{i=1}^{n+1} e^{-Y_i} \prod_{j=2}^{n+1} e^{X_j} + \sum_{i=2}^{n} e^{-X_i} + e^{-X_{n+1}} \left(1 + \eta_{n+1}\chi_{n+1}\right) \right).$$

The field redefinition $y_i = e^{-Y_i}, x_i = e^{-X_i}$ allows to rewrite the path integral as

$$\int \prod_{i=1}^{n+1} dy_i \prod_{j=2}^{n} \frac{dx_j}{x_j} \frac{dx_{n+1}}{x_{n+1}^2} d\eta_{n+1} d\chi_{n+1} \tag{3.14}$$
$$\exp\left(\sum_{i=1}^{n+1} y_i + e^t \prod_{i=1}^{n+1} y_i \prod_{j=2}^{n+1} x_j^{-1} + \sum_{i=2}^{n} x_i + x_{n+1} \left(1 + \eta_{n+1}\chi_{n+1}\right) \right).$$

Using the rescaling $\widetilde{y}_1 = y_1, \widetilde{y}_j = y_j/x_j$, for $j = 2, \ldots, n+1$ we can recast the result as

$$\int \prod_{i=1}^{n+1} d\tilde{y}_i \prod_{j=2}^{n} dx_j \frac{dx_{n+1}}{x_{n+1}} d\eta_{n+1} d\chi_{n+1} \tag{3.15}$$
$$\exp\left(\tilde{y}_1 + \sum_{i=2}^{n+1} \tilde{y}_i x_i + e^t \prod_{i=1}^{n+1} \tilde{y}_i + \sum_{i=2}^{n} x_i + x_{n+1} \left(1 + \eta_{n+1}\chi_{n+1}\right) \right)$$

By introducing the auxiliary bosonic variables u, v, we rewrite the factors $1/x_{n+1}$ in the path integral measure as follows:

$$\frac{1}{x_{n+1}} = \int du dv e^{uvx_{n+1}} \tag{3.16}$$

The integral then becomes

$$\int \prod_{i=1}^{n+1} d\tilde{y}_i \prod_{j=2}^{n+1} dx_j d\eta_{n+1} d\chi_{n+1} du dv \tag{3.17}$$
$$\exp\left(\tilde{y}_1 + \sum_{i=2}^{n+1} \tilde{y}_i x_i + e^t \prod_{i=1}^{n+1} \tilde{y}_i + \sum_{i=2}^{n} x_i + x_{n+1} \left(1 + \eta_{n+1}\chi_{n+1} + uv\right) \right)$$

that is

$$\int \prod_{i=1}^{n+1} d\tilde{y}_i \prod_{j=2}^{n+1} dx_j d\eta_{n+1} d\chi_{n+1} du dv \tag{3.18}$$
$$\exp\left(\tilde{y}_1 \left(1 + e^t \prod_{i=2}^{n+1} \tilde{y}_i \right) + \sum_{i=2}^{n} x_i(\tilde{y}_i + 1) + x_{n+1} \left(1 + \eta_{n+1}\chi_{n+1} + uv + \tilde{y}_{n+1}\right) \right)$$

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This form is convenient because the integrations over $\tilde{y}_1, x_{i=2,\dots,n+1}$ give delta functions

$$\int \prod_{i=2}^{n+1} d\tilde{y}_i du dv\, \delta\left(1+\eta_{n+1}\chi_{n+1}+uv+\tilde{y}_{n+1}\right) \prod_{i=2}^{n} \delta(\tilde{y}_i+1)\delta\left(1+e^t \prod_{i=2}^{n+1}\tilde{y}_i\right) \tag{3.19}$$

Solving the last delta function constraint in eq. (3.19) we get:

$$\tilde{y}_{n+1} = -\frac{e^{-t}}{\prod_{i=2}^{n}\tilde{y}_i}. \tag{3.20}$$

Imposing the constraints $\prod_{i=2}^{n}\delta(\tilde{y}_i+1)$ on eq. (3.20) then yields

$$\tilde{y}_{n+1} = \pm e^{-t} \tag{3.21}$$

the plus and minus signs being respectively when n is even or odd. We can then solve the last delta function appearing in (3.19) obtaining

$$1+\eta_{n+1}\chi_{n+1}+uv \pm e^{-t} = 0. \tag{3.22}$$

We have 2 bosonic variables u, v with eq. (3.22) as constraint and two fermionic coordinates. The superdimension is therefore -1 and matches the superdimension of $\mathbf{CP}^{(n|n+1)}$. So we see that the mirror geometry (apart from the sign difference in the n even and n odd cases) does not really depend on n, but only on the superdimension. So we have recast the mirror geometry in the form

$$uv + \eta\chi = a \tag{3.23}$$

in $\mathbf{C}^{(2|2)}$. The equation degenerates to $uv+\eta\chi = 0$ for $t = 0$ and n even, or $t = i\pi$ and n odd. The form of equation (3.23) is reminiscent of the deformed conifold equation

$$xy + uv = a \tag{3.24}$$

in $\mathbf{C}^4$. For this reason we will refer to equation (3.23) as the "superconifold".

We want now to compare the two conifold-like geometries. Let us begin reviewing some aspects of the geometry of the familiar conifold. The complex deformation parameter a resolves the node singularity of the conifold geometry $xy + uv = 0$, by replacing the origin with a 3-sphere. The deformed conifold is topologically T^*S^3, i.e. the cotangent bundle of a S^3. This can be seen as follows. We start by rewriting the defining equation as $\sum_{i=1}^4 x_i^2 = a$. The constant can always be taken real by suitably redefining the x_i's. Decomposing x_i into real and imaginary parts as $x_i = v_i + iw_i$, we can write equivalently

$$\sum_{i=1}^4 v_i^2 - w_i^2 = a, \quad \sum_{i=1}^4 v_i w_i = 0. \tag{3.25}$$

Interpreting w_i as coordinates along the fiber we see that the base is an S^3 with coordinates v_i's. The base of the bundle is an example of "special Lagrangian submanifold". A real middle-dimensional submanifold L of a Kähler manifold is Lagrangian if the restriction of the Kähler form on L is zero. If in addition $\mathrm{Im}\Omega_L = 0$ also holds, the submanifold is called

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special Lagrangian. Here the Kähler form on T^*S^3 can be written as $2\sum_{i=1}^4 dv_i dw_i$. This is clearly zero on the base, since $w_i = 0$. Similarly one can verify that the imaginary part of the holomorphic measure is zero when restricted to the base. Therefore the base S^3 is a special Lagrangian submanifold inside the non compact Calabi-Yau T^*S^3.

We can follow a similar analysis for $uv + \eta\chi = a$. Let us begin by rewriting equation (3.23) as

$$u_1^2 + u_2^2 + \lambda_\alpha \lambda^\alpha = a, \tag{3.26}$$

by identifying $\chi = \sqrt{2}\lambda^1$ and $\eta = \sqrt{2}\lambda^2$. We use the following decompositions into real and imaginary parts, $u_i = v_i + iw_i$ and $\lambda_\alpha = \eta_\alpha + i\nu_\alpha$. Equation (3.26) is then equivalent to

$$\sum_{i=1}^{2} v_i^2 - w_i^2 + \sum_{\alpha=1}^{2} \eta_\alpha \eta^\alpha - \nu_\alpha \nu^\alpha = a, \quad \sum_{i=1}^{2} v_i w_i + \sum_{\alpha=1}^{2} \eta_\alpha \nu^\alpha = 0. \tag{3.27}$$

We interpret (w_i, ν_α) as coordinates in the fiber and (v_i, η_α) as parameterizing the supersphere $S^{(1|2)}$, $\sum_{i=1}^2 v_i^2 + \sum_{\alpha=1}^2 \eta_\alpha \eta^\alpha = a$, in the base. Extending the notion of special Lagrangian submanifold to supermanifolds, we can ask whether $S^{(1|2)}$ is (super)special-Lagrangian. Formally then, we could view $uv + \eta\chi = a$ as $T^*S^{(1|2)}$. The standard Kähler form of $\mathbf{C}^{(2|2)}$, when expressed in terms of v_i, w_i, η, ν, is[3] $\sum_i du_i d\bar{u}_i + \sum_\alpha d\lambda_\alpha d\bar{\lambda}_\alpha = \sum_i dv_i dw_i + \sum_\alpha (d\eta_\alpha)^2 + (d\nu_\alpha)^2$ and does not reduce to zero on the base $w = \eta = 0$. We can nevertheless make a "mild" modification on the fermionic part of the Kähler form of $\mathbf{C}^{(2|2)}$ such that its restriction on the superconifold is zero. That is we consider the superconifold as embedded in a new supermanifold $\mathbf{C}_\star^{(2|2)}$ with modified Kähler form $\omega = du_i d\bar{u}_i + \epsilon_{\alpha\beta} d\lambda_\beta d\bar{\lambda}^\alpha$. The new space is still super Calabi-Yau as one can easily verify by checking that the super Monge-Ampere equation is satisfied. The new Kähler form can be further reduced to

$$\omega = -2i \sum_{i=1}^{2} dv_i dw_i - 2i \sum_{\alpha=1}^{2} d\eta_\alpha d\nu^\alpha. \tag{3.28}$$

and its restriction on $S^{(1|2)}$ is zero. Since the imaginary part of the holomorphic measure is also zero when restricted to the base, we can view $S^{(1|2)}$ as a special Lagrangian submanifold.

Another well known resolution of the ordinary conifold singularity is the so called "small resolution" which, in mathematical terms, consists in replacing the conifold with the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. In this case the origin is replaced with an S^2. We can give an explicit description as follows. We replace the singular conifold geometry $xy - uv = 0$ with the following equation

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \tag{3.29}$$

where $(z_1, z_2) \in \mathbf{CP}^1$. Since (z_1, z_2) is always different from zero, we have

$$\det \begin{pmatrix} x & u \\ v & y \end{pmatrix} = 0, \tag{3.30}$$

[3]Note that the superform $d\eta$ and $d\chi$ are commuting objects. For more about conventions on superforms I refer to section 5.

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i.e. the conifold equation. Outside the origin of $\mathbf{C}^4$, eq. (3.29) simply specifies a point in $\mathbf{CP}^1$ and therefore the new geometry coincides with the old one. At the origin instead, (z_1, z_2) are unconstrained and therefore we have a full $\mathbf{CP}^1$ which resolves the node singularity. In the supermanifold context we can proceed similarly considering the following "resolution":

$$\begin{pmatrix} u & \eta \\ \chi & v \end{pmatrix} \begin{pmatrix} z_{\text{even}} \\ z_{\text{odd}} \end{pmatrix} = 0 \tag{3.31}$$

where now $(z_{\text{even}}|z_{\text{odd}})$ lives in $\mathbf{C}^{(1|1)}/\mathbf{C}^* \equiv \mathbf{C}^{(0|1)}$. The super-conifold can be obtained from the Berezinian

$$\text{sdet} \begin{pmatrix} u & \eta \\ \chi & v \end{pmatrix} = 0. \tag{3.32}$$

Therefore in this case the singularity at the origin is replaced by $\mathbf{C}^{(0|1)}$. Note that, using the $\mathbf{C}^*$ action, $(z_{\text{even}}|z_{\text{odd}}) \sim (1|\psi)$, and that $u = -\eta\psi$ and $\chi = -v\psi$. Moreover since $\mathbf{C}^{(0|1)}$, differently from $\mathbf{CP}^1$ in the bosonic case, can be covered with only one patch, the resolution (3.31) can be globally parameterized by $(v|\,\eta, \psi)$ and therefore coincides with $\mathbf{C}^{(1|2)}$.

As a final comment let us note that the familiar conifold equation can be given a gauge invariant description in terms of four chiral superfields $(\phi_1, \phi_2, \phi_3, \phi_4)$ with $\mathrm{U}(1)$ charges $(1, 1, -1, -1)$. The gauge invariant combinations $x \equiv x_1 x_3$, $u \equiv x_1 x_4$, $v \equiv x_2 x_3$, $y \equiv x_2 x_4$ satisfy, as a constraint, the conifold equation. In the present context we would have to modify the charge assignment to $(1, 1, 1, 1)$ and therefore we do not have anymore a gauge invariant description.

4. Lagrangian submanifolds

We have seen an example of a (super)special Lagrangian in the discussion of the "super-conifold" in the last section. In the second part of the paper we want provide further interesting examples of special Lagrangians inside super-toric varieties and discuss their geometric properties.

We begin by reviewing the construction of Lagrangian submanifolds in $\mathbf{C}^n$ [15, 20, 21]. This construction will be extended to supermanifolds in the next section. We use a polar coordinate system, i.e. we parameterize $\mathbf{C}^n$ with $\{|z^i|^2, \theta^i\}$. The Kähler form for $\mathbf{C}^n$ is then

$$\omega = \sum_i d|z^i|^2 \wedge d\theta^i. \tag{4.1}$$

A Lagrangian submanifold L is a real n-dimensional subspace satisfying $\omega_{|L} = 0$, i.e. the restriction of the Kähler form on L is zero. An obvious Lagrangian is therefore θ^i =const., $\forall i$ and no constraints on the $|z^i|$'s. Let us call L_0 this Lagrangian. More interesting Lagrangians can be built out of this one. Inside L_0 we consider the subspace

$$\sum_i q_i^\alpha |z^i|^2 = c^\alpha, \ \ \alpha = 1, \ldots, n - r. \tag{4.2}$$

This is a real r-dimensional subspace of L_0. We can trade the n redundant variables $|z^i|$ for the coordinates $s^\beta, \beta = 1, \dots, r$, through the linear transformation

$$|z^i|^2 = v^i_\beta s^\beta + d^i, \ \ \beta = 1, \dots, r. \tag{4.3}$$

To satisfy eq.(4.2) we need to impose $v^i_\beta q^\alpha_i = 0$ and $q^\alpha_i d^i = c^\alpha$. Since this subspace, that we call $\mathcal{L}$, is contained in L_0 we trivially have $\omega_| = 0$ but it is not Lagrangian since it is not middle-dimensional. We can nevertheless get a Lagrangian submanifold fibering over each point of $\mathcal{L}$ a torus T^{n-r} by imposing that the angles θ^i satisfy

$$\sum_i v^i_\beta \theta^i = 0. \tag{4.4}$$

It is easy then to check that $\omega_| = 0$:

$$\omega = \sum_i d|z^i|^2 \wedge d\theta^i = \sum_{i,\beta} v^i_\beta ds^\beta \wedge d\theta^i \tag{4.5}$$

$$= \sum_\beta ds^\beta \wedge d\left(\sum_i v^i_\beta \theta^i\right). \tag{4.6}$$

Using $v^i_\beta q^\alpha_i = 0$, eq.(4.4) can be satisfied by choosing $\theta^i = q^\alpha_i \varphi_\alpha$. The angles φ_α span the torus T^{n-r}.

Consider now the Calabi-Yau $Y = \mathbf{C}^n//G$ where $G = \mathrm{U}(1)^{n-k}$ and with D-term equations

$$\sum_i Q^a_i \left|z^i\right|^2 = r^a, \quad a = 1, \dots, n-k. \tag{4.7}$$

The Calabi-Yau condition amounts to requiring $\sum_i Q^a_i = 0$, $\forall a$. The Lagrangian submanifolds of $\mathbf{C}^n$ descend to Y if the condition $v^i_\beta \theta^i = 0$ is well defined, i.e. preserved, in the quotient. The action of the a$^{\text{th}}$ U(1) group on the phase θ^i of the i$^{\text{th}}$ chiral field is $\theta_i \to \theta_i + Q^a_i \varphi^a$. Therefore, to preserve $v^i_\beta \theta^i = 0$, we need to impose

$$\sum_i Q^a_i v^i_\beta = 0. \tag{4.8}$$

Let us consider some examples.

Example 1. Consider the following locus in $\mathbf{C}^3$

$$2|z_1|^2 - |z_2|^2 - |z_3|^2 = c \tag{4.9}$$

Using $\theta^i = q^\alpha_i \varphi_\alpha$ gives $\theta_1 = 2\phi$ and $\theta_2 = \theta_3 = -\phi$. In this case we have a S^1 fibration, parameterized by ϕ, over the locus (4.9). The vectors v_β are $v_1 = (1, 1, 1)$, $v_2 = (0, 1, -1)$.

Example 2. As a second example we take in $\mathbf{C}^4$

$$2|z_1|^2 - |z_2|^2 - |z_3|^2 = c^1, \ \ |z_1|^2 - |z_4|^2 = c^2 \tag{4.10}$$

To build a Lagrangian we fiber a torus over the base (4.10) parameterized by the angles ϕ_1, ϕ_2. The condition $\theta^i = q^\alpha_i \varphi_\alpha$ yields $\theta_1 = 2\phi_1 + \phi_2$, $\theta_2 = -\phi_1$, $\theta_3 = -\phi_1$ and $\theta_4 = -\phi_2$.

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The vectors v_β are $v_1 = (1,1,1,1)$ and $v_2 = (0,1,-1,0)$. This Lagrangian will be preserved in the Kähler quotient $\mathbf{C}^4//\,\mathrm{U}(1)$ if the charges Q_i satisfy (4.8), i.e. $Q_1+Q_2+Q_3+Q_4=0$ and $Q_2 = Q_3$. Due to the first condition the quotient is automatically a Calabi-Yau manifold.

Example 3. As a final example we consider the Lagrangian (A brane)

$$|z_2|^2 - |z_4|^2 = c^1, \;\; |z_3|^2 - |z_4|^2 = c^2 \tag{4.11}$$

in the resolved conifold geometry $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P^1}$. As quotient of $\mathbf{C}^4$ this threefold is characterized by the $\mathrm{U}(1)$ charges $Q=(1,1,-1,-1)$.

All the examples considered so far are actually *special* Lagrangian submanifolds. In this context the special Lagrangian condition is equivalent to requiring $\sum_i q_i^\alpha = 0$. "A branes"in non-compact Calabi-Yau threefold like (4.11) have been studied in depth in [20, 21] where the problem of counting holomorphic instantons ending on special Lagrangian submanifolds was solved using mirror symmetry.

5. Super Lagrangian submanifolds

We now want to generalize the previous construction to toric super Calabi-Yau manifolds. The idea would be to start from constructing examples of super Lagrangians in $\mathbf{C}^{(n|m)}$ and successively study the conditions under which they descend to super Calabi-Yau's built as quotients of $\mathbf{C}^{(n|m)}$. The supermanifold $\mathbf{C}^{(n|m)}$ has Kähler potential $z^i\bar{z}^i + \psi^A\bar{\psi}^A$ and super-Kähler form

$$d|z^i|^2\wedge d\theta^i + d\psi^A d\bar{\psi}^A. \tag{5.1}$$

Our conventions for (anti-)commutations relation for superforms are as follows

$$\omega_1\omega_2 = (-1)^{a_1a_2+b_1b_2}\omega_2\omega_1 \tag{5.2}$$

where a_i and b_i are respectively the superform degree and the $\mathbf{Z}_2$ Grassmann grading of ω_i. For example dz has $a=1$ and $b=0$ while $d\psi$ has $a=b=1$. Using this rule we obtain the familiar wedge product anticommutation rule $dzd\bar{z} = -d\bar{z}dz$ but also in particular $d\psi d\bar{\psi} = d\bar{\psi}d\psi$. One should not confuse the commuting $d\psi^A$'s entering in the Kähler form with the anti-commuting $d\psi^A \equiv \partial_{\psi^A}$'s in the holomorphic measure. The d operator is $d = dz^i\partial_{z^i} + d\psi^A\partial_{\psi^A}$ with Leibnitz rule[4] $d(\omega_1\omega_2) = d\omega_1\,\omega_2 + (-1)^r\omega_1 d\omega_2$ if ω_1 is a superform of degree $a=r$.

In $\mathbf{C}^n$ the prototype for a Lagrangian submanifold is the real locus

$$\theta^i = \theta^i_0, \;\;\; i=1,\ldots,n \tag{5.3}$$

with θ^i_0 constant. Since the notion of polar coordinates does not extend to fermionic variables we need a new way to think about eq. (5.3). The Lagrangian submanifold (5.3)

[4]With this convention $\psi d\psi = -d\psi\,\psi$.

can be rewritten as $z^i = e^{2i\theta_0^i}\bar{z}^i$ and this form can be easily generalized to the supermanifold case as follows

$$z^i = e^{2i\theta_0^i}\bar{z}^i, \;\; \psi^A = e^{2i\Theta_0^A}\bar{\psi}^A. \tag{5.4}$$

This is a middle-dimensional submanifold of $\mathbf{C}^{(n|m)}$ but it fails to satisfy the condition $\omega_| = 0$. Indeed the fermionic part $d\psi^A d\bar{\psi}^A$ of the super-Kähler of $\mathbf{C}^{(n|m)}$ restricts on (5.4) to $e^{2i\Theta_0^A} d\psi^A d\psi^A \neq 0$.

A real submanifold like (5.4) becomes Lagrangian if we modify the fermionic part of ω and make it "symplectic" in the following sense:

$$\omega = i\sum_{i=1}^{n} dz^i d\bar{z}^i + i\sum_{k=1}^{m} \epsilon_{A_k\,B_k} d\psi^{A_k} d\bar{\psi}^{B_k}, \tag{5.5}$$

We will denote the corresponding space as $\mathbf{C}_\star^{(n|2m)}$. The index A_k takes the values 1, 2. Other supermanifolds will be constructed as quotients of this space. As a consequence we will then consider only supermanifolds with an even number of fermionic dimensions. With this modification the real submanifold $z^i = e^{2i\theta_0^i}\bar{z}^i$, $\psi^{A_k} = e^{2i\Theta_0^{A_k}}\bar{\psi}^{A_k}$ *is* Lagrangian since $d\psi_A d\psi^A = 0$. The new space $\mathbf{C}_\star^{(n|2m)}$ is still, obviously, super Calabi-Yau. One possible way to verify this claim is to check that the super Monge-Ampere equation $\mathrm{sdet} K_{i\bar{j}} = 1$ is satisfied:

$$\mathrm{sdet}\begin{pmatrix} \mathbf{1_{n\times n}} & & & & & \\ & 0 & 1 & & & \\ & -1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} = 1. \tag{5.6}$$

In eq. (5.6) we used the definition of superdeterminant or Berezinian:

$$\mathrm{sdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D} \tag{5.7}$$

where A, D and B, C are respectively Grassmann even and Grassmann odd matrices. We can now proceed in parallel with bosonic case considering the equation

$$q_i^\alpha |z^i|^2 + p_k^\alpha \epsilon_{A_k\,B_k}\psi^{A_k}\bar{\psi}^{B_k} = c^\alpha, \;\; \alpha = 1, \ldots, n-r. \tag{5.8}$$

We can explicitly solve eq. (5.8) for the bosonic variables $|z^i|^2$ as

$$|z^i|^2 = v_\beta^i s^\beta - r_k^i \epsilon_{A_k\,B_k}\psi^{A_k}\bar{\psi}^{B_k} + d^i \tag{5.9}$$

with the following conditions

$$q_i^\alpha v_\beta^i = 0, \;\; q_i^\alpha d_\beta^i = c^\alpha, \;\; q_i^\alpha r_k^i = p_k^\alpha. \tag{5.10}$$

The locus has real superdimension $(n-(n-r))-2m = r-2m$. Using eq. (5.8), the bosonic part of the super Kähler form gives

$$d|z^i|^2 \wedge d\theta^i = ds^\beta \wedge d(v_\beta^i\theta^i) - r_k^i \epsilon_{A_k\,B_k}(d\psi^{A_k}\bar{\psi}^{B_k} + \psi^{A_k} d\bar{\psi}^{B_k}) \wedge d\theta^i. \tag{5.11}$$

Using $\psi^{A_k} = e^{2i\Theta^k}\bar{\psi}^{A_k}$ and parameterizing the bosonic angles as $\theta^i = q^i_\alpha \phi^\alpha$ this becomes

$$- e^{2\Theta^k}\epsilon_{A_k\,B_k}(2id\Theta^k\bar{\psi}^{A_k}\bar{\psi}^{B_k} + 2d\bar{\psi}^{A_k}\,\bar{\psi}^{B_k}) \wedge d(r^i_k\theta^i). \tag{5.12}$$

The fermionic part of the Kähler form reads instead

$$ie^{2\Theta^k}\epsilon_{A_k\,B_k}(d\bar{\psi}^{A_k}d\bar{\psi}^{B_k} + 2id\Theta^k\bar{\psi}^{A_k}d\bar{\psi}^{B_k}) = -2e^{2\Theta^k}d\Theta^k\bar{\psi}^{A_k}d\bar{\psi}^{B_k} \tag{5.13}$$

where we used the property that the $d\bar{\psi}^{A_k}$'s commute. The sum of (5.12) and (5.13) is zero if we choose $r^i_k\theta^i = \Theta^k$. The Lagrangian is then a T^{n-r} fibration parametrized by $\{\phi^\alpha\}$ over the locus (5.8), with $\theta^i = q^\alpha_i\phi^\alpha,\ \Theta^k = p^\alpha_k\phi^\alpha$.

The moment map associated to the $\mathrm{U}(1)$ vector field

$$X = Q^i z^i\frac{\partial}{\partial z^i} - Q^i\bar{z}^i\frac{\partial}{\partial\bar{z}^i} + P^k\psi^{A_k}\frac{\partial}{\partial\psi^{A_k}} - P^k\bar{\psi}^{A_k}\frac{\partial}{\partial\bar{\psi}^{A_k}} \tag{5.14}$$

is

$$Q^i|z^i|^2 + P^k\epsilon_{A_k\,B_k}\psi^{A_k}\bar{\psi}^{B_k} = r \tag{5.15}$$

Note that to preserve the Kähler (5.5) form we have assigned the same charge P^k to each couple of fermionic fields ψ^{A_k}. The quotient $\mathbf{C}^{(n|2m)}_\star//\,\mathrm{U}(1)$ then is a super Calabi-Yau iff[5] $\sum_{i=1}^n Q^i = 2\sum_{k=1}^m P^k$. If we want the Lagrangian to descend to the Calabi-Yau quotient we need to preserve the constraints $v^i\theta^i = 0$ and $r^i_k\theta^i = \Theta^k$. The action of the $\mathrm{U}(1)$ group on the phases is $\theta^i \to \theta^i + Q^i_\alpha\varphi^\alpha$ and $\Theta^k \to \Theta^k + P^k_\alpha\varphi^\alpha$ and therefore we need

$$v^iQ^i = 0, \quad r^i_kQ^i = P^k. \tag{5.16}$$

The special Lagrangian condition for the submanifold (5.8) is

$$\sum_{i=1}^n q^\alpha_i - 2\sum_{k=1}^m p^\alpha_k = 0. \tag{5.17}$$

Let us consider some examples. We begin with

$$\begin{aligned} |z^1|^2 + |z^3|^2 + \epsilon_{A_1\,B_1}\psi^{A_1}\bar{\psi}^{B_1} &= c^1 \\ |z^2|^2 + |z^4|^2 + \epsilon_{A_2\,B_2}\psi^{A_2}\bar{\psi}^{B_2} &= c^2 \end{aligned} \tag{5.18}$$

in $\mathbf{C}^{(4|4)}_\star$. Note that the special Lagrangian condition is satisfied. Performing a Kähler quotient with charges $Q^i = 1,\ i = 1,\ldots,4$ and $P^k = 1,\ k = 1,2$ we obtain the super Calabi-Yau $\mathbf{CP}^{(3|4)}_\star$. One can verify that the submanifold (5.18) satisfies the conditions (5.16) and therefore descends to a special Lagrangian in $\mathbf{CP}^{(3|4)}_\star$. As a further example we can take

$$\begin{aligned} 2|z^1|^2 - |z^2|^2 - |z^4|^2 &= c^1 \\ |z^2|^2 + |z^3|^2 + \epsilon_{AB}\psi^A\bar{\psi}^B &= c^2. \end{aligned} \tag{5.19}$$

[5] More generally if we have the Kähler quotient $\mathbf{C}^{(n|2m)}_\star//\,\mathrm{U}(1)^r$ the CY condition is $\sum_{i=1}^n Q^i_\alpha = 2\sum_{k=1}^m P^k_\alpha$ with $\alpha = 1,\ldots,r$.

in the superprojective space $\mathbf{WCP}(-2,1,2,1|1,1)$ which is obtained from $\mathbf{C}_\star^{(4|2)}$ dividing by the $\mathrm{U}(1)_{\mathbf{C}}$ group with charges $(Q^i|P^k) = (-2,1,2,1|1,1)$.

Modding out by the complexified gauge group $\mathrm{U}(1)_{\mathbf{C}}$ always reduces the complex bosonic dimension by one, without changing the fermionic dimension. Since we cannot gauge away fermions we cannot have submanifolds of the form $p_k \epsilon_{A_k,B_k} \psi^{A_k} \bar{\psi}^{B_k} = c$. Therefore one additional constraint comes from requiring that, when considering the matrix of the charges

$$\left(\begin{array}{c|c} Q^i & P^k \\ q_i^\alpha & p_k^\alpha \end{array} \right),$$

the bosonic submatrix

$$\begin{pmatrix} Q^1, \dots, Q^n \\ q_1^\alpha, \dots, q_n^\alpha \end{pmatrix} \tag{5.20}$$

has maximum rank.

Let us now discuss how the special Lagrangian (5.8) map in the dual Landau-Ginzburg theory. The only novelty comes from the modified Kähler form for the fermionic directions. To learn how to proceed let us study the following bosonic gauged linear sigma model

$$\mathcal{L} = \int d^4\theta \left(i\epsilon_{AB} \bar{\Phi}^A e^{2QV} \Phi^B - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) - \frac{1}{2} \int d^2\tilde{\theta}\, t\Sigma + c.c., \quad A = 1,2. \tag{5.21}$$

It is convenient to make the field transformation

$$\begin{aligned} \Phi_1 &= \varphi_1 + i\varphi_2 \\ \Phi_2 &= \varphi_2 + i\varphi_1 \end{aligned} \tag{5.22}$$

which enables to rewrite the kinetic term for the chiral fields as $-2(\bar{\varphi}_1 e^{2QV} \varphi_1 - \bar{\varphi}_2 e^{2QV} \varphi_2)$. We now introduce the following Lagrangian:

$$\begin{aligned} \mathcal{L} = &\int d^4\theta \left(e^{2QV + B_1} - \frac{1}{2}(Y_1 + \bar{Y}_1) B_1 \right) - \int d^4\theta \left(e^{2QV + B_2} - \frac{1}{2}(Y_2 + \bar{Y}_2) B_2 \right) \\ &- \int d^4\theta \frac{1}{2e^2} \bar{\Sigma}\Sigma - \frac{1}{2} \int d^2\tilde{\theta}\, t\Sigma + c.c.. \end{aligned} \tag{5.23}$$

The equations of motion of Y_1 and Y_2 imply that

$$B_1 = \psi_1 + \bar{\psi}_1, \;\; B_2 = \psi_2 + \bar{\psi}_2 \tag{5.24}$$

where ψ_1 and ψ_2 are two chiral fields. We obtain the desired Lagrangian with the identification $\varphi_1 = e^{\psi_1}$ and $\varphi_2 = e^{\psi_2}$. Proceeding differently and integrating out the B fields gives

$$B_1 = -2QV + Log\left[-\frac{i}{2}(Y_1 + \bar{Y}_1) \right], \;\; B_2 = -2QV + Log\left[-\frac{i}{2}(Y_2 + \bar{Y}_2) \right] \tag{5.25}$$

Inserting this expression in the enlarged Lagrangian we can read off the classical dual twisted superpotential

$$\tilde{W}_{cl.} = \int d^2\tilde{\theta} Q\Sigma \,(Y_1 - Y_2 - t) \tag{5.26}$$

to which one must add the instanton correction $\tilde{W}_{inst.} = e^{-Y_1} - e^{-Y_2}$. By integrating out Σ we obtain "the dual D-term condition" $Y_1 - Y_2 = t$. The relation between the lowest components of the chiral fields φ_A and the dual twisted fields Y_A is as usual

$$\frac{1}{2}\mathrm{Re}Y_i = |\varphi_i|^2. \tag{5.27}$$

These considerations suggest that, in the fermionic generalization and after having done a field transformation similar to (5.22), the equation

$$q_i^\alpha |z^i|^2 + p_k^\alpha(|\psi_1^k|^2 - |\psi_2^k|^2) = c^\alpha \tag{5.28}$$

becomes in the dual variables

$$q_i^\alpha Y^i - p_k^\alpha(X_1^k - X_2^k) = c^\alpha. \tag{5.29}$$

The dual Landau-Ginzburg superpotential is

$$\tilde{W} = \sum_{i=1}^n e^{-Y^i} + \sum_{k=1}^m e^{-X_1^k}(1 + \eta_1^k \chi_1^k) - e^{-X_2^k}(1 + \eta_2^k \chi_2^k) \tag{5.30}$$

with D-term constraint

$$\sum_{i=1}^n Q_i Y^i + \sum_{k=1}^m Q_k(X_1^k - X_2^k) = t. \tag{5.31}$$

6. Infinitesimal deformations

In this final section we want to comment on the space of infinitesimal deformations of special Lagrangians inside a supermanifold. Let us begin by reviewing the bosonic case. There is a quite convenient way to study the local geometry of a Lagrangian in $\mathbf{C}^n$ which is familiar in symplectic geometry [25]. Locally every Lagrangian can be thought as the graph Γ_f of a closed 1 form df, where f is a smooth function from $\mathbf{R}^n$ to $\mathbf{R}$. This simply means that the Lagrangian can be seen locally as the real n-dimensional submanifold

$$\Gamma_f = \{(x^1, y^1 = \partial_{x^1} f(x^1, \ldots, x^n), \ldots, x^n, y^n = \partial_{x^n} f(x^1, \ldots, x^n)); x^1, \ldots, x^n \in \mathbf{R}\} \tag{6.1}$$

in $\mathbf{C}^n$. Indeed the restriction of the Kähler form is $k_{\Gamma_f} = \partial^2_{i,j} f\, dx^i \wedge dx^j = 0$. We would like now to understand how to impose the special Lagrangian condition in this formalism. Under the change of variables

$$z^i \to z^i = x^i + i\partial_i f(x^1, \ldots, x^n) \tag{6.2}$$

we obtain the following transformation rule for the holomorphic top form:

$$\prod_{i=1}^n dz^i = J \prod_{i=1}^n dx^i \tag{6.3}$$

where the Jacobian J is $\det(I + i\mathrm{Hess}f)$. Since $\prod_i dx^i$ is real by construction, the special Lagrangian condition, $\mathrm{Im}\Omega_{|L} = 0$, is then equivalent to

$$\mathrm{Im}\det(I + i\mathrm{Hess}f) = 0. \tag{6.4}$$

We can now study infinitesimal deformations of special Lagrangians in $\mathbf{C}^n$. Using the fact that every Lagrangian looks locally like $\mathbf{R}^n$ we can study the infinitesimal deformations of $\mathbf{R}^n$ which preserve the special Lagrangian condition. The deformation of $\mathbf{R}^n$ can be seen as the graph Γ_f, with the condition that the function f and its derivatives are infinitesimal. We can then linearize equation (6.4) to obtain

$$\mathrm{Im}\det(I + i\mathrm{Hess}f) \sim \mathrm{Tr}\,\mathrm{Hess} = \triangle f = 0. \tag{6.5}$$

This result shows that infinitesimal deformations of special Lagrangian in $\mathbf{C}^n$ are associated to harmonic functions on $\mathbf{R}^n$. Since adding a constant to f does not change Γ_f, the submanifold (6.1) is parametrized by df. Infinitesimal deformations of a special Lagrangian $\mathcal{L}$ correspond therefore to harmonic 1-forms on $\mathcal{L}$. This result is a first step toward the Mclean's theorem [26] according to which the moduli space of special Lagrangian deformations of a compact Lagrangian L is a smooth manifold of dimension $b^1(L)$.

We can now discuss the extension to the super Lagrangian case. We consider for simplicity $\mathbf{C}_\star^{(n|2)}$. Using the decomposition $z^i = x^i + iy^i,\ \psi^A = \eta^A + i\chi^A$, the Kähler form $\sum_{i=1}^n idz^i d\bar{z}^i + \sum_{k=1}^m i\epsilon_{A\,B} d\psi^A d\bar{\psi}^B$ becomes

$$\omega = 2\sum_{i=1}^n dx^i dy^i + 2\sum_{k=1}^m d\eta_A d\chi^A. \tag{6.6}$$

The natural generalization of (6.1) is

$$\Gamma = \{z^i = x^i + i\partial_{x^i} f(x,\eta),\, \psi^A = \eta^A + ig^A(x,\eta)\} \tag{6.7}$$

The restriction of the Kähler on this locus turns out to be

$$2\frac{\partial^2 f}{\partial x^m \partial x^n} dx^m \wedge dx^n + 2dx^m d\eta^A \left(\frac{\partial^2 f}{\partial x^m \partial \eta^A} + \frac{\partial g_A}{\partial x^m}\right) + 2d\eta^A d\eta^B \frac{\partial g_A}{\partial \eta^B}. \tag{6.8}$$

Requiring $k_\Gamma = 0$ yields

$$g_A = -\frac{\partial f}{\partial \eta^A}, \qquad \frac{\partial g^A}{\partial \eta^B} = \delta^A_B h(x). \tag{6.9}$$

These conditions imply that $g^A = \eta^A h(x)$ and $f = f_0(x) - \frac{1}{2}\eta^A \eta_A h(x)$. The top holomorphic form is

$$\prod_{i,A} dz^i d\psi^A = \mathcal{J} \prod_{i,A} dx^i d\eta^A \tag{6.10}$$

where $\mathcal{J}$ is the super-Jacobian

$$\mathcal{J} = \mathrm{sdet}\begin{pmatrix} 1 + i\mathrm{H}ess f & -i\partial^2 f/\partial x^m \partial \eta_A \\ i\partial^2 f/\partial x^m \partial \eta^A & \delta^A_B(1 + ih) \end{pmatrix} \tag{6.11}$$

To study local deformations we specialize to the Lagrangian $z^i = e^{2i\theta_0^i}\bar{z}^i$, $\psi^A = e^{2i\Theta_0^A}\bar{\psi}^A$ in $\mathbf{C}_\star^{(n|2)}$. A Lagrangian which differs from this one by an infinitesimal deformation looks then locally like (6.7), with the condition that f and its derivatives are kept small. To require that the deformation is special Lagrangian we need to impose $\mathrm{Im}\mathcal{J} = 0$ which, to linear order in the deformation, is equivalent to

$$\mathrm{Im}\,\frac{\det(1 + i\mathrm{Hess}f)}{\det[\delta_B^A(1 + ih)]} \sim \triangle f - h = 0, \tag{6.12}$$

where, as before, $\triangle$ is the ordinary Laplacian in $\mathbf{R}^n$. The last equation splits into

$$\triangle f_0 = h, \quad \triangle h = 0. \tag{6.13}$$

This suggests that special Lagrangian deformations are associated to a pair of harmonic functions h and f_0^h, the second being a solution of the homogeneous equation for f_0. Extrapolating this result we would expect a moduli space of dimension $b_1(L)^2$ for compact special Lagrangians. One can easily extend this result to Lagrangian submanifolds in $\mathbf{C}_\star^{(n|m)}$.

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